INDIAN STATISTICAL INSTITUTE

Probability Theory I: B. Math (Hons.) I

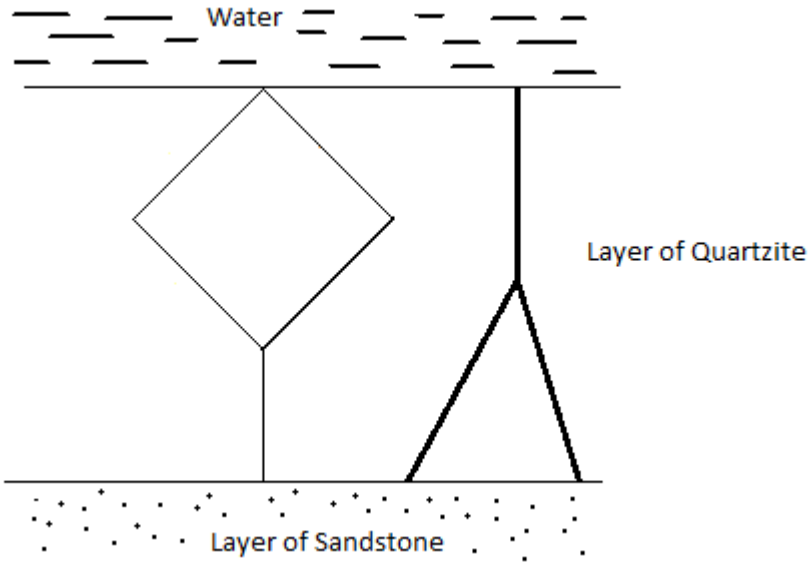
Semester I, Academic Year 2018-19

Semestral Exam

Date: 14/11/2018 **Total Marks: 50** **Duration: 10 am - 1 pm**

- **Show all your work and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. Consider the drainage network model as described in the class with each path open with probability $p \in (0,1)$ independently of others. Suppose one particular such model consists of two disjoint components with r and s many paths (please see the diagram below in the special case $r = 5$ and $s = 3$) out of which exactly X and Y are open, respectively.



Compute the following with full justification (for $r, s \in \mathbb{N}$).

(a) (5 marks) The conditional distribution of X given $Z := X + Y$.
(b) (5 marks) $E(X|Z)$
(c) (10 marks) $Var(X|Z)$.

Please turn over to the other side.

2. (10 marks) Suppose r distinguishable balls are arranged at random in $n\,(\geq 2)$ boxes B_1, B_2, ..., B_n. For each i, let us denote by X_i the number of balls in B_i. Fix $m \in \mathbb{N}$ with $m < n$. Calculate the expectation of the following random variable

$$\frac{\sum\limits_{1\leq i<j\leq m} X_i X_j}{\sum\limits_{1\leq i<j\leq n} X_i X_j}.$$

3. (5 + 5 =10 marks) Recall the *top-to-random shuffle* of a pack of 52 cards as described in the class. Let N be the number of shuffles needed for the initial bottommost card to come to the top. Calculate $E(N)$ and $Var(N)$.

4. Suppose that the initial number M of bacteria in a bacteria colony follows binomial distribution with parameters $n = 100$ and $p = 0.5$. Assume that the bacteria behave independently of each other, and each of them die within an hour with probability 0.8 independently of M. Let Z denote the number of surviving bacteria in the colony after one hour.

   (a) (5 marks) Calculate $E(s^Z|M)$, where $s \in \mathbb{R}$.

   (b) (5 marks) Using (a) or otherwise, find the probability mass function of Z.